Fourier Transform.

Here we consider two underlying spaces, $\mathbb{R}^n$ and $\mathbb{T}^n$.

- $\mathbb{R}^n$ is locally cpct Abelian group under addition, and Lebesgue measure $m = dx$ is invariant under action of $\mathbb{R}^n$ by itself.

- $\mathbb{T}^n = S^1 \times \ldots \times S^1$, where $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ (or $S^1 = \{z \in \mathbb{C} : |z| = 1\}$). $\mathbb{T}^n$ is a compact Abelian grp under addition (multiplication). The Lebesgue measure on $\mathbb{R}^n$ descends to $\mathbb{T}^n$ (as arclength on $S^1 \subseteq \mathbb{C}$) as an invariant measure.

A <u>character</u> of (locally) cpct Abelian grp w/ an invariant measure is a $\mathbb{C}$-valued, measurable fcn φ s.t. $|\varphi|=1$ and $\varphi(x+y)=\varphi(x)\varphi(y)$.

<u>Thm1</u>. The characters on $\mathbb{R}^n$ and $\mathbb{T}^n$ are $\varphi(x)=e^{2\pi i \xi\cdot x}$, $\xi\in\mathbb{R}^n$ or $\mathbb{Z}^n$ respectively.

$\xi\cdot x=\sum_{j=1}^n \xi_j x_j$

See Folland for pf.

1 <u>Periodic functions / fcns on $\mathbb{T}^n$</u>.

<u>Recall</u>. $H=L^2(\mathbb{T}^n,m)=L^2(\mathbb{T}^n)$ is a Hilbert space w/ $\langle f,g\rangle=\int_{\mathbb{T}^n} f(x)\overline{g(x)}\,dx$ as inner product; $\|f\|_{L^2}=\langle f,f\rangle^{1/2}$.

Thm 2. The characters $\left\{ E_k(x) = e^{2\pi i k\cdot x} \right\}_{k\in\mathbb{Z}^n}$ form an orthonormal (ON) basis for $L^2(\mathbb{T}^n)$.

Pf. Use Stone-Weierstrass to deduce that $\{E_k\}$ is dense in $\mathcal{C}(\mathbb{T}^n)$ w/ $\|\cdot\|_\infty$. Since m Radon, $\mathbb{T}^n$ cpt, $\{E_k\}$ is dense in $\mathcal{C}(\mathbb{T}^n)$ wrt $\|\cdot\|_{L^p}$-norm $(1\le p<\infty)$, and since $\mathcal{C}(\mathbb{T}^n)$ is dense in $L^p(\mathbb{T}^n)$, $\{E_k\}$ is dense in $L^p(\mathbb{T}^n)$. $\boxtimes$

ON is a simple calculation.

By Hilbert space theory, we can write

- $$f(x) = \sum_{k\in\mathbb{Z}^n} \hat{f}(k)\, e^{2\pi i k\cdot x}, \text{ where}$$
- $$\hat{f}(k) = \langle f, E_k\rangle = \int_{\mathbb{T}^n} f(x) e^{-2\pi i k\cdot x}\, dx$$

and the series on RHS above converges to f in $L^2(\mathbb{T}^n)$.

$\{\hat{f}(k)\}_{k\in\mathbb{Z}^n}$ are called <u>Fourier coeff's</u> and $\hat{f}:\mathbb{Z}^n\to\mathbb{C}$ is called <u>Fourier transform</u> of f.

Also, by <u>Parseval's identity</u>,

- $$\|f\|_{L^2(\mathbb{T}^n)} = \|\hat{f}\|_{L^2(\mathbb{Z}^n)} = \Big(\sum_{k\in\mathbb{Z}^n} |\hat{f}(k)|^2\Big)^{1/2}.$$

Also, $\ell^2 = \ell^2(\mathbb{Z}^n)$ w/ norm

<u>Hausdorff-Young Ineq.</u> Let $1\le p\le 2$, and let $q=p^*$ $\left(\frac{1}{p}+\frac{1}{q}=1\right)$. If $f\in L^p(\mathbb{T}^n)$, then $\hat{f}\in \ell^q(\mathbb{Z}^n)$ and

$$\|\hat{f}\|_{\ell^q(\mathbb{Z}^n)} \le \|f\|_{L^p(\mathbb{T}^n)}$$

<u>Pf.</u> We use Riesz-Thorin Interpolation w/ endpoints $(p,q)=(1,\infty)$, $(p,q)=(2,2)$.

<u>$(p,q)=(1,\infty)$</u>: $|\hat{f}(k)| \le \int_{\mathbb{T}^n} |f| = \|f\|_{L^1(\mathbb{T}^n)}$

$$\Rightarrow \|\hat{f}\|_{\ell^\infty} \le \|f\|_{L^1}.$$

$\underline{(p,q)=(2,2)}$. Parseval $\Rightarrow \|\hat{f}\|_{\ell^2} = \|f\|_{L^2}$.

RT Interpolation $\Rightarrow$ concl. of HY Ineq. $\blacksquare$

(2) Fourier transform on $L^p(\mathbb{R}^n)$.

First, assume $f \in L^1(\mathbb{R}^n)$ and define

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i \xi \cdot x}\, dx$$

By previous results, $\hat{f} \in C(\mathbb{R}^n)$ and clearly $\|\hat{f}\|_{L^\infty} = \|\hat{f}\|_u \le \|f\|_{L^1}$. Thus, we have the <u>Fourier transform</u>

$\mathcal{F} : L^1(\mathbb{R}^n) \to BC(\mathbb{R}^n)$, bdd linear operator.

We would like to have HY for $\mathcal{F}$ on $\mathbb{R}^n$, so that $\mathcal{F} : L^p(\mathbb{R}^n) \to L^{p^*}(\mathbb{R}^n)$ is bdd linear operator for $1 \le p \le 2$.

To use RI as before, we need the other endpoint $(p,q)=(2,2)$. This requires more work than for $X=\mathbb{T}^n$.

<u>Basic Prop's of $\mathcal{F}$.</u> Let $f,g\in L^1(\mathbb{R}^n)$.

(a) $(\tau_y f)^\wedge(\xi)=e^{-2\pi i\xi\cdot y}\hat{f}(\xi)$ and

$$(\tau_\eta \hat{f})(\xi)=\left(e^{2\pi i\eta\cdot x}f(x)\right)^\wedge(\xi).$$

(b) If $T\in GL(\mathbb{R}^n)$ then

$$(f\circ T)^\wedge=|\det T|^{-1}\hat{f}\circ(T^*)^{-1}$$

$T^*:\mathbb{R}^n\to\mathbb{R}^n$ adjoint of T.

(c) $(f*g)^\wedge=\hat{f}\hat{g}$.

(d) If $x^\alpha f\in L^1$ for $|\alpha|\le k$, then $\hat{f}\in C^k$ and $\partial^\alpha\hat{f}=((-2\pi i x)^\alpha f)^\wedge$.

(e) If $f \in C^k$, $\partial^\alpha f \in L^1$ for $|\alpha| \le k$, and $\partial^\beta f \in C_0$ for $|\beta| \le k-1$, then $(\partial^\alpha f)^\wedge = (2\pi i \xi)^\alpha \hat{f}$ for $|\alpha| \le k$.

(f) (Riemann – Lebesgue's Lemma). $\hat{f} \in C_0(\mathbb{R}^n)$.

Pf. (a)–(d) are formally easy. Pfs are DIY.

(e) Suffices to prove for $k=1$, $\alpha = (0,\dots,0,1)$. Thus, $\partial_n f \in L^1$, $f \in L^1 \cap C_0$. Then,

$$(\partial_n f)^\wedge(\xi) = \int_{\mathbb{R}^n} (\partial_n f)(x) e^{-2\pi i \xi \cdot x}\, dx = \left\{ \begin{matrix} \text{Fubini} \\ x = (x', x_n) \end{matrix} \right\}$$

$$= \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^{\infty} (\partial_n f)(x', x_n) e^{-2\pi i \xi_n x_n}\, dx_n \right) e^{-2\pi i \xi' \cdot x'}\, dx'$$

$$= \{\text{IBP}\} = \int_{\mathbb{R}^{n-1}} \left(\left[f(x_n, x') e^{-2\pi i x_n \xi_n} \right]_{-\infty}^{\infty} - \right.$$

$$\left. \int_{-\infty}^{\infty} f(x', x_n) \left[-(2\pi i \xi_n) \right] e^{-2\pi i \xi_n x_n}\, dx_n \right) e^{-2\pi i \xi' \cdot x'}\, dx'$$

$= \left\{ f \in \mathcal{C}_0 \Rightarrow \lim_{x_n \to \infty} f(x', x_n) = 0 \right\} = 2\pi i \xi_n \hat{f}(\xi).$

(f) If $f \in C_c^\infty$, then $\partial^\alpha f \in \mathcal{C}_c^\infty$ for all $\alpha \Rightarrow \xi^\alpha \hat{f} \in BC \Rightarrow |\xi|^{2k} \hat{f} \in BC$ and $\| |\xi|^{2k} \hat{f} \|_u \le \| \Delta^k f \|_{L^1} \Rightarrow |\hat{f}(\xi)| \le C |\xi|^{-2k} \to 0$ as $|\xi| \to \infty$ $(k \ge 1) \Rightarrow \hat{f} \in \mathcal{C}_0$.

Since $\mathcal{C}_c^\infty$ is dense in L^1, it follows that for any $f \in L^1$ $\exists$ $f_n \in \mathcal{C}_c^\infty$ s.t. $\|f - f_n\|_{L^1} \to 0 \Rightarrow \|\hat{f} - \hat{f}_n\|_u \to 0$.

Since $\mathcal{C}_0$ w/ $\|\cdot\|_u$ is Banach (complete) $\Rightarrow \hat{f} \in \mathcal{C}_0$. ∎

<u>Recall</u> $\mathcal{S} \subseteq \mathcal{C}^\infty$, Schwartz space, is a Frechet space w/ $\{\|\cdot\|_{(N,\alpha)}\}_{(N,\alpha) \in \mathbb{Z}_+^{n+1}}$ and

$$\|f\|_{(N,\alpha)} = \sup_{x \in \mathbb{R}^n} (1 + |x|)^N |\partial^\alpha f(x)|$$

<u>Thm 3</u> $\mathcal{F}$ is continuous linear map

$\mathcal{S} \to \mathcal{S}$.

<u>Pf.</u> We first note that, by Basic Prop's (d), $\hat{f} \in C^\infty$ and $\partial^\alpha \hat{f} = \left((-2\pi i x)^\alpha f\right)^\wedge$

$$\Rightarrow |\partial^\alpha \hat{f}(\xi)| \le C \| x^\alpha f \|_{L^1}$$

$(2\pi)^{|\alpha|}$

$$\le C \| (1+|x|)^{|\alpha|} f \|_{L^1}$$

$$\le C' \| (1+|x|)^{|\alpha|+n+1} f \|_u$$

$$c \int \frac{1}{(1+|x|)^{n+1}} dx < \infty$$

$$\Rightarrow \| \partial^\alpha \hat{f} \|_{(0,\alpha)} \le C \| f \|_{(|\alpha|+n+1, 0)}$$

Next note that $|\xi|^k \le C_N \sum_{j=1}^n |\xi_j|^k \Rightarrow$

$$(1+|\xi|)^N |\partial^\alpha \hat{f}(\xi)| \le C \sum_{k=0}^N \sum_{j=1}^n |\xi_j|^k |\partial^\alpha \hat{f}(\xi)|$$

$$= C' \sum_{k,j} |\xi_j^k \partial^\alpha \hat{f}(\xi)| \le$$

$$C'' \sum_{j,k} |(\partial_j^k (x^\alpha f))^\wedge(\xi)| \leq$$

$$C'' \sum_{j,k} \| \partial_j^k (x^\alpha f) \|_{L^1} \leq$$

$$C'' \sum_{\substack{|\beta| \leq |\alpha| \\ |\gamma| \leq N+n+1}} \| x^\beta \partial^\gamma f \|_{L^1} \leq$$

$$C'' \sum_{\substack{M \leq N+n+1 \\ |\beta| \leq |\alpha|}} \| f \|_{(M,\beta)} .$$

This $\Rightarrow$ $\hat{f} \in \mathcal{S}$ and $f \to \hat{f}$ is cont. in Fréchet space $\mathcal{S}$. ▨

To establish $\mathcal{F} : L^2 \to L^2$ we shall need the <u>inverse Fourier transform</u>.

For $g \in L^1(\mathbb{R}^n)$,

$$g^\vee(x) = \int_{\mathbb{R}^n} g(\xi)\, e^{2\pi i \xi \cdot x}\, d\xi$$

Note that $g^\vee(x) = \hat{g}(-x)$.

<u>Fourier Inversion Thm.</u> If $f, \hat{f} \in L^1$, then $f = f_0$ a.e., $f_0 \in C_0$, and

$$f = (\hat{f})^\vee = (f^\vee)^\wedge = f_0.$$

<u>Pf.</u> The proof hinges on <u>Plancherel's formula</u>: If $f, g \in L^1$, then

$$\int \hat{f}(\xi) g(\xi)\, d\xi = \int f(x) \hat{g}(x)\, dx.$$

<u>Pf.</u> This is simply an application of Fubini once you note $\hat{f}g, f\hat{g} \in L^1$.

The idea of pf for FI thus comes from the formal "calculation".

$$(\hat{f})^{\vee}(x) = \int \hat{f}(\xi)\, e^{2\pi i \xi \cdot x}\, d\xi$$

$$= \int\Big(\int f(y)\, e^{2\pi i \xi\cdot(x-y)}\, dy\Big) d\xi$$

$$\sim \int f(y) \Big(\int e^{-2\pi i \xi\cdot(y-x)}\, d\xi\Big) dy$$

$$\sim \int f(y)\, \hat{1}(y-x)\, dy$$

Thus, $(\hat{f})^{\vee}(x) = f(x)$ if $\hat{1} = \delta_0$ where δ_0 = Dirac Delta measure at 0. Of course, this makes no sense (at this point).

Instead we will use an approximate identity.

__Lemma 1.__ Let $\varphi(x) = e^{-\pi|x|^2}$. Then,

$$\hat{\varphi}(\xi) = e^{-\pi|\xi|^2} = \varphi(\xi).$$

__Pf.__ Suffices to prove for $n=1$, since

$$e^{-\pi|x|^2} = e^{-\pi(x_1^2+\cdots+x_n^2)} = \varphi(x_1)\cdots\varphi(x_n),$$

so $\hat{\varphi}(\xi) = \hat{\varphi}(\xi_1)\cdots\hat{\varphi}(\xi_n)$. For $n=1$, we note that $|x|^2 = x^2$ and

$$\varphi'(x) = -2\pi x\,\varphi(x) = -i\,(-2\pi i x\,\varphi(x))$$

By FT and BP, we then get

$$2\pi i\xi\,\hat{\varphi}(\xi) = -i\,\hat{\varphi}'(\xi) \quad \text{or}$$

$$\hat{\varphi}'(\xi) = 2\pi\xi\,\hat{\varphi}(\xi)$$

Thus, $\varphi, \hat{\varphi}$ satisfy same 1st order linear ODE. Moreover, $\varphi(0) = 1$ and $\hat{\varphi}(0) = \int e^{-\pi x^2}\,dx = 1 \Rightarrow \hat{\varphi}(\xi) = \varphi(x)$ by ODE theory. ☒

Consider, for $t>0$ and $x \in \mathbb{R}^n$,

$$\varphi(\xi) = e^{2\pi i \xi\cdot x} e^{-\pi|t\xi|^2},$$

By BP, $\hat{\varphi}(y) = \tau_x(\psi(t\xi))^\wedge(y)$

$$= \tau_x \frac{1}{t^n}\hat{\psi}(y/t) = \psi_t(y-x).$$

Since $\int \psi = 1$, we have for f as in FI thm, $f * \psi_t \to f$ a.e., and, moreover by Plancherel,

$$(f*\psi_t)(x) = \int f(y)\varphi_t(y-x)\,dy$$

$$= \int f(y)\hat{\varphi}(y)\,dy = \int \hat{f}(\xi)\varphi(\xi)\,d\xi$$

$$= \int \hat{f}(\xi) e^{2\pi i\xi\cdot x} e^{-\pi t^2|\xi|^2}\,d\xi$$

$$\longrightarrow \int \hat{f}(\xi) e^{2\pi i \xi\cdot x}\,d\xi \text{ as } t\to 0^+$$

by Dominated Convergence ($\hat{f}\in L^1$).

In other words, if we set $f_0(x) = \int \hat{f}(\xi) e^{2\pi i \xi \cdot x} \, d\xi = (\hat{f})^{\vee}(x)$, then $f_0 \in C_0$ (by Riemann-Lebesgue) and $f_0 = f$ a.e. $\boxtimes$

<u>Remarks</u>: As a consequence, $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is an isomorphism. We already proved $f \in \mathcal{S} \Rightarrow \hat{f} \in \mathcal{S}$ and since $(f^{\vee})(x) = \hat{f}(-x)$ it is clear that $f \in \mathcal{S} \Rightarrow f^{\vee} \in \mathcal{S}$. By FI Thm, $f = (\hat{f}^{\vee})^{\wedge} = (\hat{f})^{\vee}$ which shows $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is isomorphism.

Plancherel's Thm. If $f \in L^1 \cap L^2$, then $\hat{f} \in L^2$. Moreover, if $f, g \in L^1 \cap L^2$, then $\langle \hat{f}, \hat{g} \rangle = \langle f, g \rangle$.

Cor1. $\mathcal{F}$ extends as a unitary map $L^2 \to L^2$.

Pf of Cor1. $L^1 \cap L^2$ is dense in L^2 (as it contains in particular $\mathcal{S}$). $\blacksquare$

Pf of P Thm. Since $\mathcal{S}$ is dense in L^2 and $\mathcal{F}$ is isomorphism $\mathcal{S} \to \mathcal{S}$, it suffices to prove the identity:

$$\int \hat{f}\bar{\hat{g}} = \int f\bar{g}, \quad \forall f, g \in \mathcal{S}.$$

This follows immediatly from the previously established P. formula ($f \in L^1$) and FI Thm. Recall $\hat{h}(-\xi) = (h^\vee)(\xi)$. It is also obvious that $\overline{\hat{h}(\xi)} = (\bar{h})^\wedge(-\xi)$. Using the notation τ for the involution $(\tau h)(x) = h(-x)$ and $\bar{h}(x) = \overline{h(x)}$, we have

$$\int \hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi = \int \hat{f}(\xi)\big(\tau(\bar{g})^\wedge\big)(\xi)\,d\xi$$

$$= \int \hat{f}(\xi)(\bar{g}^\vee)(\xi)\,d\xi = \{\text{P. formula}\}$$

$$= \int f(x)\,(\bar{g}^\vee)^\wedge(x)\,dx = \{\text{FI Thm}\}$$

$$= \int f(x)\overline{g(x)}\,dx. \quad \blacksquare$$

Cor 1 establishes the other endpoint $(p,q)=(2,2)$ for the use of the Riesz-Thorin Interp. Thm. We obtain the HY ineq. as in the compact case $X=\mathbb{T}^n$.

<u>Hausdorff-Young Inequality</u> $(X=\mathbb{R}^n)$

Suppose $1\le p\le 2$, $q=p^*$ $\left(\frac{1}{p}+\frac{1}{q}=1\right)$ and $f\in L^p$. Then $\hat{f}\in L^q$ and

$$\|\hat{f}\|_{L^q}\le\|f\|_{L^p}.$$

Poisson Summation Formula.

If $f \in L^1(\mathbb{T}^n)$, then it corresponds to a periodic function $F : \mathbb{R}^n \to \mathbb{C}$ ($F(x) = F(x-k)$, $\forall k \in \mathbb{Z}^n$. (Of course, F will not be in $L^1(\mathbb{R}^n)$.))

If you start w/ $F \in L^1(\mathbb{R}^n)$ you can try to create a periodic function $f : \mathbb{T}^n \to \mathbb{C}$ by

- $\sum_{k \in \mathbb{Z}^n} f(x-k)$, or
- $\sum_{k \in \mathbb{Z}^n} \hat{f}(k) e^{2\pi i k \cdot x}$.

One would need to make sense out of the infinite series, though.

Poisson Summation Formula

Let $f \in C(\mathbb{R}^n)$ and assume that $|f(x)| \le \frac{c}{(1+|x|)^{n+\alpha}}$, $|\hat{f}(\xi)| \le \frac{c}{(1+|\xi|)^{n+\alpha}}$,

for some $\alpha > 0$. Then,

$$(*) \quad \sum_{k \in \mathbb{Z}^n} f(x-k) = \sum_{\kappa \in \mathbb{Z}^n} \hat{f}(\kappa) e^{2\pi i \kappa \cdot x}$$

and both series converge absolutely and uniformly on $\mathbb{T}^n$.

Pf. Since $\sum_k \frac{1}{(1+|k|)^{n+\alpha}} < \infty$, it is clear that both sides of $(*)$ converge abs. + unif. for $x \in \overline{Q}$, where $Q = [0,1)^n \subseteq \mathbb{R}^n$ which can be identified w/ $\mathbb{T}^n$.

For conv. of $\sum f(x-k)$, $x \in Q$, we use $|f(x-k)| \leq \frac{C}{(1+|x-k|)^{n+\alpha}}$

$\Rightarrow$ if $|k| \geq 2\sqrt{n}$, say, then

$|f(x-k)| \leq \frac{C}{(|k|-\sqrt{n})^{n+\alpha}}$ and

$$\sum_{|k| \geq 2\sqrt{n}} \frac{1}{(|k|-\sqrt{n})^{n+\alpha}} < \infty.$$

Thus, both sides of (*) define periodic continuous fcns on $\mathbb{R}^n$ or, equivalently, functions

$P_1 f = \text{LHS}$, $P_2 f = \text{RHS}$ in $\mathcal{C}(\mathbb{T}^n)$.

$$(P_1 f)^\wedge(\varkappa) = \int_Q (P_1 f)(x)\, e^{-2\pi i \varkappa \cdot x}\, dx$$

$$\sum_k \int_Q f(x-k)\, e^{-2\pi i \varkappa \cdot x}\, dx = \left\{ \begin{matrix} x' = x-k \\ dx' = dx \\ e^{-2\pi i \varkappa \cdot k} = 1 \end{matrix} \right\}$$

$$= \sum_k \int_{Q-k} f(x) e^{-2\pi i k \cdot x} dx = \int_{\mathbb{R}^n} f(x) e^{-2\pi i k \cdot x} dx$$

$$= \hat{f}(k).$$

Thus, $P_2 f$ is Fourier series of $P_1 f$. Since $P_1 f \in C(\mathbb{T}^n) \subseteq L^2(\mathbb{T}^n)$, the series defining $P_2 f$ converges to $P_1 f$ in L^2. Since the series also converges pt-wise to $P_2 f \in C(\mathbb{T}^n)$, we conclude that $P_1 f = P_2 f$ as desired. $\boxtimes$